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THE COUPLED DYNAMIC PROBLEM OF THERMOELASTICITY FOR A HALF-SPACE†

M. V. DOLOTOV and I. D. KILL'

Moscow

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An exact solution in closed form of the coupled dynamic problem of thermoelasticity is obtained for a half-space with a boundary condition of the first kind. The normal stress, perpendicular to the free surface, in the neighbourhood of the elastic wave front is investigated. Copyright © 1996 Elsevier Science Ltd.

A fairly complete list of publications on coupled dynamic problems of thermoelasticity can be found in [1]. Analytic methods of solving the problems described in [1] are effective mainly for small values of the dimensionless time of the thermal action. Nevertheless, even high values of dimensionless time correspond to physically short dimensional times. For such values the small-parameter method, which is widely used, leads to fairly cumbersome solutions.

The plane coupled problem of thermoelasticity for a half-space with a finite velocity of heat propagation and concentrated thermal action was considered in [2]. The assumptions made when inverting the Laplace transform are not sufficiently justified.

1. An elastic half-space $z \geqslant 0$ is at rest at an absolute temperature $T = T_0$ up to the instant of time $t = 0$. When $t = 0$ the temperature of the boundary $z = 0$ of the half-space is increased to a value T_C and then remains constant. It is required to determine the absolute temperature T and the stress in the half-space, taking the dynamic components and the coupling into account.

We will introduce the following dimensionless quantities

$$T' = \frac{T - T_0}{T_C - T_0}, \quad \sigma' = \frac{(1-2\nu)\sigma_{zz}}{\alpha E(T_C - T_0)}, \quad z' = \frac{cz}{a}, \quad t' = \frac{c^2 t}{a} \tag{1.1}$$

where c is the velocity of longitudinal elastic waves, α is the coefficient of linear expansion, E is Young's modulus, ν is Poisson's ratio, and a is the thermal diffusivity of the material. The primes on the dimensionless quantities will henceforth be omitted.

To determine $T(z, t)$ and $\sigma(z, t)$ we need to solve the following boundary-valued problem [1]

$$\frac{\partial^2\sigma}{\partial z^2} - \frac{\partial^2\sigma}{\partial t^2} = \frac{\partial^2 T}{\partial t^2}, \quad \frac{\partial^2 T}{\partial z^2} = (1+\mu^2)\frac{\partial T}{\partial t} + \mu^2\frac{\partial\sigma}{\partial t}$$

$$T(0,t) = 1, \quad \sigma(0,t) = 0, \quad T(z,0) = \frac{\partial T}{\partial t}(z,0) = \sigma(z,0) = \frac{\partial\sigma}{\partial t}(z,0) = 0, \quad |T| < \infty, \quad |\sigma| < \infty \tag{1.2}$$

$$\mu^2 = \frac{(1+\nu)E}{(1-\nu)(1-2\nu)}\frac{\alpha^2 T_0 a}{k}$$

where μ^2 is the coupling parameter and k is the thermal conductivity.

The solution of boundary-value problem (1.2) can be found by means of a Laplace transformation with respect to t. The transforms $T^*(z, s)$, $\sigma^*(z, s)$ of the required functions have the form [1]

$$T^*(z,s) = \frac{(\lambda_1^2 - s^2)e^{-\lambda_1 z} - (\lambda_2^2 - s^2)e^{-\lambda_2 z}}{s^2 r}, \quad \sigma^*(z,s) = \frac{e^{-\lambda_1 z} - e^{-\lambda_2 z}}{r} \tag{1.3}$$

$$r = \sqrt{(s-s_1)(s-s_2)}, \quad s_{1,2} = 1 - \mu^2 \pm 2i\mu, \quad \lambda_{1,2} = \sqrt{s(s+1+\mu^2 \pm r)/2}$$

$$\arg r = \arg\lambda_1 = \arg\lambda_2 = 0 \quad \text{for} \quad s > 1 - \mu^2$$

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We will demonstrate the procedure for inverting the transforms (1.3) using the example of $\sigma^*(z, s)$. We put

$$\sigma^*(z,s)=\sigma_1^*(z,s)-\sigma_2^*(z,s),\quad \sigma_k^*=e^{-\lambda_k z}/r,\quad k=1,2 \tag{1.4}$$

In the complex s plane with cuts on the negative part of the real axis and the section $[s_1, s_2]$, the functions r, λ_1, λ_2, σ_1^*, σ_2^* can be separated into single-valued branches, defined by the initial choice of the arguments in (1.3). These branches λ_1 and λ_2 satisfy the relation

$$\lambda_1=s+O(1),\quad \lambda_2=\sqrt{s}+O(1/\sqrt{s}),\quad s\to\infty$$
$$\lambda_1=s+o(1),\quad \lambda_2=\sqrt{s}+o(\sqrt{s}),\quad s\to 0 \tag{1.5}$$

By the inversion theorem

$$\sigma(z,t)=\sigma_1(z,t)-\sigma_2(z,t),\quad \sigma_k(z,t)=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\sigma_k^*(z,s)e^{st}ds,\quad k=1,2;\quad \omega>1-\mu^2 \tag{1.6}$$

We will consider a closed contour in the complex s plane formed by arcs of the circles $|s| = R$, $|s| = \rho$, $|s - s_k| = \rho$ ($k = 1, 2$) the edges of the cuts $\operatorname{Im} s = 0$, $-\infty < \operatorname{Re} s < -\rho$, $\operatorname{Re} s = 1 - \mu^2$, $-2\mu + \rho < \operatorname{Im} s < 2\mu - \rho$ and the straight line $\operatorname{Re} s = \omega$. The function $\sigma_2^*(z, s)e^{st}$ on the arc $|s| = R$, by virtue of the second relation in (1.5), satisfies the conditions of Jordan's lemma. Using Cauchy's integral theorem and taking the limit as $R \to \infty$, $\rho - 0$ we can reduce the integral $\sigma_2(z, t)$ from (1.6) to the sum of integrals over the edges of the cuts (the integrals over the circles of radius ρ vanish in the limit).

By calculating the values of the radicals on the edges of the cuts using formulae from [3], taking into account the chosen branches r and λ_2, we obtain after reduction

$$\sigma_2(z,t)=-A(z,t,\mu^2)+e^{(1-\mu^2)t}B(z,t,\mu^2) \tag{1.7}$$

$$A(z,t,\mu^2)=\frac{2}{\pi}\int_0^\infty\frac{ye^{-y^2 t}}{q}\sin pz\,dy,\quad B(z,t,\mu^2)=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}e^{-zu_+}\cos(u_-z-2\mu t\cos y)dy$$

$$q=\sqrt{y^4+2(1-\mu^2)y^2+(1+\mu^2)^2},\quad p=y\sqrt{(1+\mu^2-y^2+q)/2}$$

$$u_\pm=\{\pm(1-\mu^2)(1-\mu\sin y)\mp 2\mu^2\cos^2 y+(1+\mu^2-2\mu\sin y)\sqrt{1+\mu^2+2\mu\sin y}\}^{1/2}/\sqrt{2}$$

When $t < z$, the function $\sigma_1^*(z, s)e^{st}$, by virtue of the first relation in (1.5), satisfies the conditions of Jordan's lemma on the arc $|s| = R$, $\operatorname{Re} s > \omega$. Using the contour formed by this arc and the section of the straight line $\operatorname{Re} s = \omega$ and taking the limit as $R \to \infty$, we obtain $\sigma_1(z, t) = 0$ ($t < z$).

If $t > z$, then $\sigma_1^*(z, s)e^{st}$ satisfies the conditions of Jordan's lemma on the arc $|s| = R$, $\operatorname{Re} s < \omega$. Using the same contour as when inverting $\sigma_2^*(z, s)$ we obtain $\sigma_1(z, t) = e^{t(1-\mu^2)}B(z, t > z)$ (the sum of the integrals over the edges of the horizontal section is equal to zero).

The inversion of $T^*(z, s)$ is carried out in the same way. Note that in this case the limit of the integral over the circle $|s| = \rho$ is unity.

We finally obtain

$$T(z,t)=1-\frac{1}{\pi}\int_0^\infty\frac{1+\mu^2+q+y^2}{yq}e^{-y^2 t}\sin pz\,dy-$$
$$-\frac{\mu e^{(1-\mu^2)t}}{\pi}\int_{-\pi/2}^{\pi/2}e^{-u_+z}\{[(1-\mu^2)(\mu-\sin y)-2\mu\cos^2 y]\cos(u_-z-2\mu t\cos y)- \tag{1.8}$$
$$-(1+\mu^2-2\mu\sin y)\cos y\sin(u_-z-2\mu t\cos y)\}\frac{dy}{(1-\mu^2)^2+4\mu^2\cos^2 y}[1-\eta(t-z)]$$
$$\sigma(z,t)=A(z,t,\mu^2)-e^{t(1-\mu^2)}B(z,t,\mu^2)[1-\eta(t-z)]$$

where $\eta(x)$ is the Heaviside unit function.

A direct check shows that expressions (1.8) are in fact the solution of boundary-value problem (1.2). Note that when $\mu = 0$, Eqs (1.8) becomes the well-known solution of the uncoupled dynamic problem [1].

2. The integrals in (1.8) for small values of z and t can be evaluated by expanding them in series in μ^2. The corresponding solution is identical with the solution obtained by the small-parameter method [1].

Note that, by virtue of (1.1), large values of the dimensionless time t correspond to even extremely small values of dimensional time. For large values of z and t the small-parameter method is not very effective due to the need to retain a considerable number of complex terms.

A computer calculation of $A(z, t, \mu^2)$, $B(z, t, \mu^2)$ for large z and t is difficult due to the considerable oscillations of the integrands.

We will confine ourselves to investigating the stress $\sigma(z, t)$ for large values of z and t in the neighbourhood of the elastic wave front.

For an approximate calculation of $B(z, t, \mu^2)$ we will use a method which is essentially an extension of the "short-time" method [1].

Using the relation obtained in Section 1

$$L_s\{e^{t(1-\mu^2)}B(z,t,\mu^2)\eta(t-z)\} = e^{-\lambda_1 z}/r$$

where L_s is the Laplace transformation operator, and the displacement theorem, we obtain

$$L_s\{B(z,t,\mu^2)\eta(t-z)\} = e^{-\lambda z}/\sqrt{s^2+4\mu^2}, \quad \lambda(s,\mu^2) = \lambda_1(s+1-\mu^2,\mu^2) \tag{2.1}$$

Retaining the first three terms in the Laurent expansion of the function $\lambda(s, \mu^2)$ in the neighbourhood of $s = \infty$

$$\lambda(s,\mu^2) = s + \left(1-\frac{\mu^2}{2}\right) + \frac{\mu^2(4-\mu^2)}{8s} - \frac{\mu^4(2+\mu^2)}{16s^2} + \ldots \tag{2.2}$$

we substitute the approximate value of $\lambda(s, \mu^2)$ thereby obtained into (2.1). Then using the binomial expansion and the delay theorem, we have

$$B^{(0)}(z,t,\mu^2)\eta(t-z) = L_t^{-1}\left\{\frac{e^{-z(s+\alpha_0+\alpha_1/s)}}{\sqrt{s^2+4\mu^2}}\right\} = e^{-\alpha_0 z}L_{t-z}^{-1}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!\mu^{2n}}{(n!)^2}\frac{e^{-\alpha_1 z/s}}{s^{2\eta+1}}\right\} \tag{2.3}$$

where L_t^{-1} is the inverse operator to L_s. The series converges absolutely when $|s| > 2\mu$. Using theorem 29.3 of [4] in (2.3) it is possible to transfer to the originals term by term. Doing this, using formula 5.5(40) of [5], we obtain

$$B^{(0)}(z,t,\mu^2) = e^{-\alpha_0 z}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}\frac{\mu^{2n}(t-z)^n}{\alpha_1^n z^n}J_{2n}(2\sqrt{\alpha_1 z(t-z)}), \quad t \geqslant z \tag{2.4}$$

where $J_k(x)$ is the Bessel function of the first kind. The series converges when $t \geqslant z$. Using the inequality $|J_k(\zeta)| \leqslant |\zeta/2|^k$ [6], it can be shown that the series in (2.4) converges and defines the function $B^{(0)}(z, t, \mu^2)$ that is regular in t for all values of t.

Note that when $t < z$ it is more convenient to use the following series for the calculations

$$B^{(0)}(z,t,\mu^2) = e^{-\alpha_0 z}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}\frac{\mu^{2n}(z-t)^n}{\alpha_1^n z^n}I_{2n}(2\sqrt{\alpha_1 z(z-t)}) \tag{2.5}$$

($I_k(x)$ is the modified Bessel function of the first kind), obtained from (2.4) using well-known relations connecting $J_k(x)$ and $I_k(x)$.

We will estimate the function $F(z, t) = B(z, t, \mu^2) - B(0)(z, t, \mu^2)$. Applying the prediction theorem ([4], p. 40) and theorems on differentiation and on the initial value of the original to the original $F(z, t)\eta(t - z)$, we arrive at relations from which, using (2.2), we can successively find

$$\begin{gathered} F(z,z)=0, \quad \frac{\partial F}{\partial t}(z,z)=0, \quad \frac{\partial^2 F}{\partial t^2}(z,z) = \frac{\mu^4(2+\mu^2)}{16}ze^{-z(1-\mu^2/2)} \\ F(z,t) = \frac{\mu^4(2+\mu^2)}{32}ze^{-z(1-\mu^2/2)}(t-z)^2+\ldots \end{gathered} \tag{2.6}$$

Consequently

$$F(z,t) = O(\mu^4), \quad \mu^2 \to 0; \quad F(z,t) = O((z-t)^2), \quad z \to t$$

Note that the order of the error with respect to μ^2 and $z - t$ is not changed if we take as the approximate value of $B(z, t, \mu^2)$ the sum of the first two terms of series (2.5).

It follows from (2.8), (2.5) and (2.6), in particular, that the values of the jump in the stress $\sigma(z, t)$ at the elastic wave front is equal to $-\exp(\mu^2 z/2)$, which agrees with the results obtained in (1).

We will estimate the integral $A(z, t, \mu^2)$. Consider the function $\Phi(w)$, which differs from $A(z, t, \mu^2)$ by having $\sqrt{(y^4 + 2(1-w)y^2 + (1+\mu^2)^2)}$ instead of q. Then

$$A(z, t, \mu^2) = \Phi(\mu^2) \tag{2.7}$$

The quantities $\Phi(-\mu^2)$ and $\Phi'(-\mu^2)$ are expressed in terms of elementary functions of the errors using relations 1.4(15) and 2.5(26) of [5], and formally obtained from them by differentiation with respect to the parameter. Using the asymptotic expression of the error function for large values of the argument [7], it can be shown that when $\sqrt{t} - z/(2\sqrt{t}) \gg 1$ the inequality $\Phi'(-\mu^2) < 0$ holds. From this relation and (2.7) we obtain, for sufficiently small μ^2

$$A(z,t,\mu^2) \leqslant \Phi(-\mu^2) = \frac{1}{2}e^{t\mu_1^2}\left[e^{-z\mu_1}\operatorname{erfc}\left(\mu_1\sqrt{t} - \frac{\mu_1 z}{2\sqrt{t}}\right) - e^{z\mu_1}\operatorname{erfc}\left(\mu_1\sqrt{t} + \frac{\mu_1 z}{2\sqrt{t}}\right)\right], \quad \mu_1^2 = 1+\mu^2 \tag{2.8}$$

Hence, for sufficiently small μ^2 and large t $\Phi(-\mu^2)$ is the majorant of $A(z, t, \mu^2)$.

The contribution of $A(z, t, \mu^2)$ to $\sigma(z, t)$ on the right semi-neighbourhood of the elastic wave front $z = t$ is small for the values of μ^2 and t considered. A calculation for $\mu^2 = 1.14 \times 10^{-2}$ (steel) with $z_0 = t_0 = 10$ (which corresponds to a distance of the elastic wave front from the free surface of 2×10^{-8} m) gives $A(z, t, \mu^2) \leqslant 9.4 \times 10^{-3}$, whereas $e^{(1-\mu^2)t}B(z, t, \mu^2) = 0.9446$. Consequently, for $z = t_0 + 0$ the term $A(z, t, \mu^2)$ is not larger than 1.1% of the solution (1.8). For large values of t_0 the contribution to the solution of $A(z, t, \mu^2)$ is reduced.

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